FINITE UNIONS OF BALLS IN $\mathbb{C}^n$ ARE RATIONALLY CONVEX

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– Is this a prologue, or the posy of a ring?
– 'Tis brief, my lord.

Hamlet, III:2

1. A compact set $K \subset \mathbb{C}^n$ is called *polynomially convex* if for any point $z \notin K$ there exists a polynomial P such that $|P(z)| > \max_{\xi \in K} |P(\xi)|$. Replacing polynomials by rational functions, one gets the definition of a *rationally convex* compact set. These notions are interesting, in particular, because any function holomorphic in a neighbourhood of a polynomially (respectively, rationally) convex set can be uniformly on this set approximated by polynomials (respectively, rational functions).

An old problem asks whether any finite union of disjoint closed balls in $\mathbb{C}^n$ is polynomially convex. It is known only that the answer is positive for at most three balls [2]. In this note, we show that the *rational* convexity of any such union follows almost immediately from the results of Julien Duval and Nessim Sibony [1].

Theorem. *Any union of finitely many disjoint closed balls in $\mathbb{C}^n$ is rationally convex.*

Note that it follows from the construction of the examples in [2] and [3] and the argument principle that this statement is false for polydiscs and complex ellipsoids in $\mathbb{C}^3$.

2. Let us first recall that according to Theorem 1.1 in [1], *if ω is a non-negative d-closed $(1,1)$-form on $\mathbb{C}^n$ such that $\mathbb{C}^n \setminus \operatorname{supp}\omega$ is relatively compact in $\mathbb{C}^n$, then for any $s > 0$, the set $\{z \in \mathbb{C}^n \mid \operatorname{dist}(z, \operatorname{supp}\omega) \geq s\}$ is rationally convex.* We shall need the following corollary of this result.

Proposition. *Let φ be a strictly plurisubharmonic function on an open subset $U \subset \mathbb{C}^n$ such that its Levi form $dd^c\varphi$ extends to a positive d-closed $(1,1)$-form on the whole $\mathbb{C}^n$. If the set $K_\varphi = \{z \in U \mid \varphi(z) \leq 0\}$ is compact, then it is rationally convex.*

Proof. Fix a small $\varepsilon > 0$ and consider a smooth convex non-decreasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(t) \equiv 0$ for $t \leq \varepsilon$, $f'(t) > 0$ for $\varepsilon < t < 2\varepsilon$, and $f'(t) \equiv 1$ for $t \geq 2\varepsilon$. Note that

$$dd^c f(\varphi) = f'(\varphi)dd^c\varphi + f''(\varphi)d\varphi \wedge d^c\varphi \geq f'(\varphi)dd^c\varphi.$$

Hence, if we set

$$\omega_\varepsilon = \begin{cases} dd^c\varphi & \text{on } \mathbb{C}^n \setminus \{\varphi \leq 2\varepsilon\}, \\ dd^c f(\varphi) & \text{on } \{\varphi \leq 2\varepsilon\}, \end{cases}$$

then ω_ε satisfies the conditions of the Duval–Sibony theorem and its support $\operatorname{supp}\omega_\varepsilon$ is precisely $\mathbb{C}^n \setminus \{\varphi < \varepsilon\}$. As ε and $s = s(\varepsilon)$ can be chosen arbitrarily close to zero, we see that K_φ is the intersection of rationally convex sets and hence rationally convex itself. □

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Remarks. 1° It follows from other results in [1] that any rationally convex compact set in $\mathbb{C}^n$ has a fundamental system of neighbourhoods of the form $\{\varphi < 0\}$, where φ satisfies the assumptions of the proposition.

2° For comparison, note that a compact set $K \subset \mathbb{C}^n$ is polynomially convex if and only if it has a fundamental system of neighbourhoods of the form $\{\varphi < 0\}$, where now φ is an exhausting strictly plurisubharmonic function on the whole $\mathbb{C}^n$.

3. We can now prove the theorem. Let $\overline{B}(a_j, r_j) = \{z \in \mathbb{C}^n \mid \|z - a_j\|^2 \leq r_j^2\}$, $j = 1, \ldots, N$, be a collection of pairwise disjoint closed balls. In a neighbourhood of their union $\bigsqcup \overline{B}(a_j, r_j)$, consider the function φ that is equal to

$$\varphi_j(z) = \|z - a_j\|^2 - r_j^2$$

near each $\overline{B}(a_j, r_j)$. Then $K_\varphi = \bigsqcup \overline{B}(a_j, r_j)$ and

$$dd^c\varphi = \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\overline{z}_k$$

is the standard flat Kähler form on $\mathbb{C}^n$. Hence, $\bigsqcup \overline{B}(a_j, r_j)$ is rationally convex by the proposition from §2. □

References

[1] J. Duval, N. Sibony, *Polynomial convexity, rational convexity, and currents*, Duke Math. J. **79** (1995), 487–513.

[2] E. Kallin, *Polynomial convexity: The three spheres problem*, Proceedings of the conference on complex analysis (Minneapolis, Minn., 1964), pp. 301–304, Springer, Berlin, 1965.

[3] A. M. Kytmanov, G. Khudaiberganov, *Example of a non-polynomially convex compactum consisting of three nonintersecting ellipsoids*, Sibirsk. Mat. Zh. **25**:5 (1984), 196–198.

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